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Complex modes and frequencies in damped structural vibrations

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Abstract

It is demonstrated that a state space formulation of the equation of motion of damped structural elements like cables and beams leads to a symmetric eigenvalue problem if the stiffness and damping operators are self-adjoint, and that this is typically the case in the absence of gyroscopic forces. The corresponding theory of complex modal analysis of continuous systems is developed and illustrated in relation to optimal damping and impulse response of cables and beams with discrete viscous dampers.

1. Introduction

The equation of motion of structural elements such as cables, beams, plates, etc. can typically be expressed in the form

$$\mathscr{D}_x u(x,t) + m\ddot{u}(x,t) = f(x,t), \tag{1}$$

where $u(x,t)$ is the displacement at the location x at time t. $f(x,t)$ is the force corresponding to the displacement $u(x,t)$. A prime denotes differentiation with respect to the spatial co-ordinate x, and a dot differentiation with respect to time t. $\mathscr{D}_x$ denotes an even order self-adjoint differential spatial operator, representing the stiffness of the structural element. Simple examples are a taut cable with axial force T, for which $\mathscr{D}_x u = -(Tu')'$, and a beam with bending stiffness EI, for which $\mathscr{D}_x u = (EIu'')''$. In addition to the self-adjoint character of the stiffness differential operator $\mathscr{D}_x$, it will be assumed that there is no energy transfer through the boundaries. The case of dampers at the boundaries can be treated by an equivalent system with the dampers located just inside the boundary.

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Vibration properties of the structural element are often studied by assuming the existence of free vibrations of the form

$$u(x,t) = u_n(x)\,\mathrm{e}^{\lambda_n t}, \tag{2}$$

where the parameter λ is determined from the eigenvalue problem obtained by substitution of Eq. (2) into Eq. (1),

$$(\mathscr{D}_x + \lambda_n^2 m)u_n(x) = 0. \tag{3}$$

For a self-adjoint stiffness operator, and boundary conditions without energy transport, the eigenvalue problem is symmetric with real-valued eigenfunctions $u_n(x,t)$ and eigenvalues λ_n^2. If Eq. (3) is multiplied by $u_n(x)$ and integrated over the element it follows from positive elastic and kinetic energy that $\lambda_n^2 \leqslant 0$, $n = 1, 2, \ldots$, and thus $\lambda_n = \mathrm{i}\omega_n$, where $\omega_n \geqslant 0$ are the real-valued natural vibration frequencies.

In a linear problem governed by a differential equation it follows from energy considerations, that damping must be proportional to the time derivative of the displacement function $u(x,t)$. In the following this dependence will be expressed via the damping operator $\mathscr{C}_x\dot{u}$. The occurrence of the first time derivative in the equation of motion leads to the occurrence of terms with the linear factor λ_n in the eigenvalue problem corresponding to Eq. (3). It turns out that insight into the structure of the problem as well as efficient means of analysis can be found by recasting the quadratic eigenvalue problem of damped vibrations into an expanded linear form. For discrete systems this approach dates back to work by Foss [1], and it is now standard procedure for numerical solution techniques, see e.g., Ref. [2]. However, the general formulation for continuous systems is still under development, see e.g., Refs. [3–5] treating special cases of isolated dampers on beams, and Yang and Wu [6] who developed a general but rather elaborate procedure for non-symmetric eigenvalue problems.

In the present paper a general formulation is developed, from which it appears, that many typical damping mechanisms are described by self-adjoint damping operators $\mathscr{C}_x$, whereby the corresponding damped eigenvalue problem can be cast into an expanded symmetric form. The expanded system is obtained by introducing an independent representation of the momentum, like in the Hamiltonian formulation of mechanics. The symmetry of the expanded eigenvalue problem leads to a fairly straightforward extension of the classical theory of modal analysis of undamped vibrations, based on an expansion of the solution in mode shapes. The general theory is illustrated by examples of optimal damping of cables and beams with isolated dampers. Modal loads are derived for concentrated and uniformly distributed loading, and the effect for impulsive load is illustrated.

2. State space format

In a system described by a linear differential equation damping must be proportional to the time derivative $\dot{u}(x,t)$ to insure dissipation of energy. The equation of motion can therefore be written in the form

$$\mathscr{D}_x u(x,t) + \mathscr{C}_x \dot{u}(x,t) + m\ddot{u}(x,t) = f(x,t), \tag{4}$$

where $\mathscr{C}_x$ is a spatial operator that represents the damping forces, and may include spatial derivatives, delta functions, etc.

Modal decomposition of the dynamic response depends on orthogonality relations for the individual terms in the equation of motion. In order to obtain a linear eigenvalue formulation and the corresponding pair of orthogonality relations for the general damped problem the momentum $m\dot{u}(x,t)$ is represented by an independent variable according to the definition

$$mv(x,t) = m\dot{u}(x,t). \tag{5}$$

Clearly this defines the new independent variable as $v(x,t) = \dot{u}(x,t)$, but definition (5) is used to emphasize the background in the independent representation of momentum as in Hamiltonian mechanics.

When the inertial force in the equation of motion is represented by the momentum variable, the damped equation of motion and the definition of the momentum variable can be combined into the state space equation format

$$\begin{bmatrix} \mathscr{D}_x & 0 \\ 0 & -m \end{bmatrix} \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} + \begin{bmatrix} \mathscr{C}_x & m \\ m & 0 \end{bmatrix} \begin{bmatrix} \dot{u}(x,t) \\ \dot{v}(x,t) \end{bmatrix} = \begin{bmatrix} f(x,t) \\ 0 \end{bmatrix}. \tag{6}$$

In contrast to the original equation of motion, that is of second order in time, the state space equation is of first order in time, similar in form to that of damped discrete systems. The symmetry of the state space format is obtained because the second equation is written as a momentum equation, and because the damping term is retained in its original form $\mathscr{C}_x\dot{u}$, i.e., as a time derivative of the displacement. Eq. (6) will be self-adjoint with respect to the state space vector $[u(x,t), v(x,t)]$ provided both the stiffness operator $\mathscr{D}_x$ and the damping operator $\mathscr{C}_x$ are self-adjoint.

It is interesting to observe that if the state space equations are written in evolution form, i.e., as an equation for the time derivative of the state space vector, the symmetry of the system does not appear explicitly, because the damping term will have to be expressed in terms of $v(x,t)$ instead of $\dot{u}(x,t)$. This has sometimes lead to the belief that damping would by itself lead to non-adjoint system properties.

2.1. Complex modes and eigenvalues

Vibration modes and the corresponding frequencies are found by assuming free vibrations of the form (2). The exponential form of the time variation leads to the following state vector mode representation:

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} u_n \\ \lambda_n u_n \end{bmatrix}, \tag{7}$$

where the relation $v_n = \lambda_n u_n$ clearly suggests the redundancy of the formulation, introduced to make the corresponding eigenvalue problem linear.

Substitution of relations (2) and (7) into the state space equation (6) leads to the linear eigenvalue problem

$$\left(\begin{bmatrix} \mathscr{D}_x & 0 \\ 0 & -m \end{bmatrix} + \lambda_n \begin{bmatrix} \mathscr{C}_x & m \\ m & 0 \end{bmatrix} \right) \begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8}$$

When both stiffness and damping operator are self-adjoint, the eigenvalue problem is symmetric, and thus there is no need to distinguish between left and right eigenvectors, as in the analysis of Yang and Wu [6]. The common forms of viscous damping leads to self-adjoint damping operator as illustrated by the examples.

In the present context it will be assumed that there is no transport of energy over the boundaries, and that springs and dampers at the boundaries are included right inside the boundaries in the operators $\mathscr{D}_x$ and $\mathscr{C}_x$. Multiplication of the eigenvalue equation (8) by $[u_r(x), v_r(x)]$ and use of the symmetry leads to the two orthogonality relations

$$\int_L [u_r(x)\mathscr{D}_x u_n(x) - v_r(x) m v_n(x)]\,\mathrm{d}x = A_n \delta_{rn}, \tag{9}$$

and

$$\int_L [u_r(x)\mathscr{C}_x u_n(x) + v_r(x) m u_n(x) + u_r(x) m v_n(x)]\,\mathrm{d}x = B_n \delta_{rn}, \tag{10}$$

where A_n and B_n are normalizing constants. Upon introduction of $v_n = \lambda_n u_n$ the following orthogonality relations for $u_n(x)$ are obtained:

$$\int_L u_r(x)[\mathscr{D}_x - \lambda_r \lambda_n m] u_n(x)\,\mathrm{d}x = A_n \delta_{rn}, \tag{11}$$

$$\int_L u_r(x)[\mathscr{C}_x + (\lambda_r + \lambda_n) m] u_n(x)\,\mathrm{d}x = B_n \delta_{rn}. \tag{12}$$

The normalizing constants A_n and B_n are not independent, and it appears to be advantageous to replace them with a single constant M_n, generalizing the classic modal mass of undamped modes. When adding λ_n times (12) to Eq. (11) for $r = n$ the integrand contains the terms corresponding to the homogeneous equation of motion (4), and thus $A_n + \lambda_n B_n = 0$. Both normalizing constants can therefore be replaced by a generalized modal mass M_n via the substitutions

$$A_n = -\lambda_n B_n = -2\lambda_n^2 M_n. \tag{13}$$

It then follows from Eqs. (11) and (12) that the generalized modal mass is given by either of the two expressions,

$$\begin{aligned} M_n &= \int_L u_n(x) \left[m + \frac{1}{2\lambda_n} \mathscr{C}_x \right] u_n(x)\,\mathrm{d}x \\ &= \frac{1}{2} \int_L u_n(x) \left[m - \frac{1}{\lambda_n^2} \mathscr{D}_x \right] u_n(x)\,\mathrm{d}x. \end{aligned} \tag{14}$$

The first of these expressions clearly shows M_n as an extension of the classic modal mass, while the second is the average of the classic modal mass and a generalization of the classic modal stiffness. Furthermore, as the mode shapes appear directly and not in the Hermitean conjugate form, the modes $u_n(x)$ can always be normalized to make the corresponding generalized modal mass M_n real.

2.2. Modal equations of motion

Modal analysis is carried out by representing the solution as an expansion of orthogonal modes. In the state space formulation this amounts to an expansion of the form

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \sum_n r_n(t) \begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix}, \tag{15}$$

where $r_n(t)$ is a generally complex amplitude function. Substitution of this representation into the state space equation (6) and use of the orthogonality relations (9) and (10) gives the first order modal equations of motion

$$A_n r_n(t) + B_n \dot{r}_n(t) = f_n(t), \tag{16}$$

with modal loads defined as

$$f_n(t) = \int_L u_n(x) f(x,t)\,\mathrm{d}x. \tag{17}$$

Introduction of the generalized modal mass from Eq. (13) gives the normalized form of the modal equations of motion,

$$\dot{r}_n(t) - \lambda_n r_n(t) = \frac{1}{2\lambda_n M_n} f_n(t). \tag{18}$$

This is the first order state space modal equation equivalent of the classic second order modal equation.

The definition of the modal forces by Eq. (17) implies that they appear as coefficients in the following formal expansions of the state space force vector $[f(x,t), 0]$:

$$\begin{bmatrix} f(x,t) \\ 0 \end{bmatrix} = \sum_n \frac{f_n(t)}{A_n} \begin{bmatrix} \mathscr{D}_x & 0 \\ 0 & -m \end{bmatrix} \begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix} = \sum_n \frac{f_n(t)}{B_n} \begin{bmatrix} \mathscr{C}_x & m \\ m & 0 \end{bmatrix} \begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix}. \tag{19}$$

The modes and eigenvalues occur in complex conjugate pairs. In the case of zero damping the contributions to the second component are pure imaginary, and therefore the momentum equation corresponding to the second component is identically satisfied also for a truncated system including a finite set of mode pairs. However, this is not guaranteed in the case of general damping.

The initial conditions of the modal equations are found by using the modal state space representation (15) at time $t = 0$,

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \sum_n r_n(0) \begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix}. \tag{20}$$

The coefficients $r_n(0)$ are obtained by use of the A- or B-orthogonality relations (9) or (10), whereby

$$\begin{aligned} r_n(0) &= \frac{1}{A_n}\int_L [u_n(x)\mathscr{D}_x u(x,0) - v_n(x) m v(x,0)]\,\mathrm{d}x \\ &= \frac{1}{B_n}\int_L [u_n(x)\mathscr{C}_x u(x,0) + v_n(x) m u(x,0) + u_n(x) m v(x,0)]\,\mathrm{d}x. \end{aligned} \tag{21}$$

These relations are expressed in terms of the mode shape $u_n(x)$ and the generalized modal mass M_n as

$$\begin{aligned} r_n(0) &= \frac{-1}{2\lambda_n^2 M_n}\int_L u_n(x)[\mathscr{D}_x u(x,0) - \lambda_n m v(x,0)]\,\mathrm{d}x \\ &= \frac{1}{2\lambda_n M_n}\int_L u_n(x)[(\mathscr{C}_x + \lambda_n m) u(x,0) + m v(x,0)]\,\mathrm{d}x, \end{aligned} \tag{22}$$

where the dependence on initial velocity $v(x,0)$ is the same, while two expressions are available for the dependence on initial position $u(x,0)$. In the undamped case information about the initial position is stored in the real part of $r_n(0)$ and information about the initial velocity in the imaginary part. In the case of general damping the effects are mixed.

3. Examples

The following examples illustrate complex mode analysis of simple cable and beam problems involving concentrated viscous dampers. In these problems the effect of the damper is closely linked to the complex character of the modes. Optimal damping of a mode is obtained for an intermediate magnitude of the damping constant that is sufficiently small to permit motion and at the same time sufficiently large to lead to substantial energy dissipation. The examples illustrate the similarity of the analytical technique for cables and beams.

3.1. Taut cable with damper

Fig. 1 shows a taut cable of length ℓ fixed by rigid supports and with a concentrated damper with parameter c at $x = a$. The objective is to find the complex modes and their damping properties, and thereby enable selection of optimum damper properties. In addition the analysis

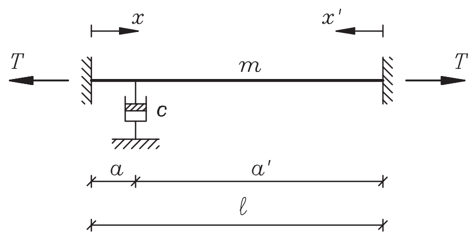

Fig. 1. Taut cable with external viscous damper.

includes the modal loads for an arbitrarily located impact force. The optimal damping problem was treated numerically by Pacheco et al. [7] while Krenk [8] and Main and Jones [9] used complex modes. The more general problem of cables with sag has been treated via complex modes by Krenk and Nielsen [10], and the nonlinear problem of whirling motion of cables with dampers by Nielsen and Krenk [11].

The equation of motion of the cable is

$$Tu'' - m\ddot{u} + f(x,t) = c\dot{u}\delta(x-a), \tag{23}$$

where T is the cable force, and m is the mass per unit length. This is a special case of the general equation (4) with $\mathscr{D}_x u = -(Tu')'$, while the right side of the equation represents the concentrated damping force, corresponding to the self-adjoint damping operator

$$\mathscr{C}_x \dot{u}(x,t) = c\dot{u}(x,t)\delta(x-a). \tag{24}$$

Thus, the problem belongs to the class of symmetric damped vibration problems treated in Section 2.

3.1.1. Homogeneous solution

When solving a differential equation with a discontinuity at $x = a$ it is convenient to consider the homogeneous equation

$$Tu'' - m\ddot{u} = 0, \quad x \neq a, \tag{25}$$

on both sides of the discontinuity. At the location of the damper the solution satisfies the discontinuity equation

$$T(u'_{a+} - u'_{a-}) = c\dot{u}. \tag{26}$$

The solution for free vibrations is expressed in the form

$$u(x,t) = u_n(x)\exp(\mathrm{i}\omega_n t), \tag{27}$$

where ω_n is the generally complex modal frequency. For free vibrations the modal damping ratio ζ_n is determined from the complex modal frequency ω_n via the representation

$$\omega_n = |\omega_n|\left(\sqrt{1-\zeta_n^2} + \mathrm{i}\zeta_n\right). \tag{28}$$

Substitution of the modal representation (27) into the homogeneous equation (25) leads to the mode shape

$$u_n(x) = \begin{cases} u_n(a)\dfrac{\sin(\beta_n x)}{\sin(\beta_n a)}, & 0 \leqslant x \leqslant a, \\[2ex] u_n(a)\dfrac{\sin(\beta_n x')}{\sin(\beta_n a')}, & 0 \leqslant x' \leqslant a', \end{cases} \tag{29}$$

where the primed parameters refer to the right part of the cable as shown in Fig. 1. The parameter β_n is the wave number, related to the frequency by

$$\beta_n = \omega_n\sqrt{\frac{m}{T}}. \tag{30}$$

The frequency is seen to be proportional to the wave number in this case.

3.1.2. Complex eigenfrequencies

The wave number is determined from the discontinuity equation (26), describing the action of the damper. Substitution of the homogeneous solution (28) into the discontinuity condition (26) gives

$$\cot(\beta_n a) + \cot(\beta_n a') = -\mathrm{i}\frac{c}{\sqrt{mT}}. \tag{31}$$

The damping parameter c is seen to appear in the equation via the non-dimensional damping coefficient

$$\eta = \frac{c}{\sqrt{mT}}. \tag{32}$$

It is convenient to rewrite Eq. (32) in a form, where the special case of undamped vibrations appear as the left side of the equation. By use of the substitution $a' = \ell - a$ and rearrangement the equation is obtained in the form

$$\tan(\beta\ell) = \frac{\mathrm{i}\eta\sin^2(\beta a)}{1 + \mathrm{i}\eta\cos(\beta a)\sin(\beta a)}. \tag{33}$$

An iterative solution is obtained from the undamped wave numbers $\beta_{n,0} = n\pi/\ell$, $n = 1, 2, \ldots$ in a couple of iterations for small values of a/ℓ. If the damper is close to an anti-node of the undamped vibration mode different modes of vibration may occur as discussed by Main and Jones [9], and the simple iterative method (33) may experience convergence problems.

For $na \ll \ell$ the equation has a simple asymptotic solution, Krenk [8]. It is obtained by using single term expansions for all the trigonometric functions in Eq. (33). When using the notation $\beta_n = \beta_{n,0} + \Delta\beta_n$, the asymptotic form of Eq. (33) can be written as

$$\frac{\Delta\beta_n}{\beta_n} \simeq \frac{a}{\ell}\frac{\mathrm{i}\eta\beta_n a}{1 + \mathrm{i}\eta\beta_n a}. \tag{34}$$

As $a \ll \ell$ the magnitude of the wave number increment $|\Delta\beta_n|$ is small relative to the original undamped value $\beta_{n,0}$. It is then permissible to use $\beta_{n,0}$ instead of β_n in relation (34), which becomes an explicit formula for the wave number increment $\Delta\beta_n$.

The damping is small, and the damping ratio can then be extracted from the complex wave number representation as

$$\zeta_n = \frac{\mathrm{Im}[\omega_n]}{|\omega_n|} = \frac{\mathrm{Im}[\beta_n]}{|\beta_n|} \simeq \frac{\mathrm{Im}[\Delta\beta_n]}{\beta_{n,0}}. \tag{35}$$

The asymptotic damping ratio then follows from (34) as

$$\frac{\zeta_n}{a/\ell} \simeq \frac{\eta\beta_{n,0}a}{1 + (\eta\beta_{n,0}a)^2} = \frac{\eta n\pi a/\ell}{1 + (\eta n\pi a/\ell)^2}. \tag{36}$$

This relation identifies a universal relation between the normalized modal damping ratio $\zeta_n\ell/a$ and the damping, represented by $\eta n\pi a/\ell = \eta a\beta_{n,0}$. The curve is shown in Fig. 2. It follows from Eq. (36) that optimal damping corresponds to

$$\eta_{opt} = \frac{1}{\pi n a/\ell}, \quad \zeta_{n,opt} \simeq \frac{a}{2\ell}. \tag{37}$$

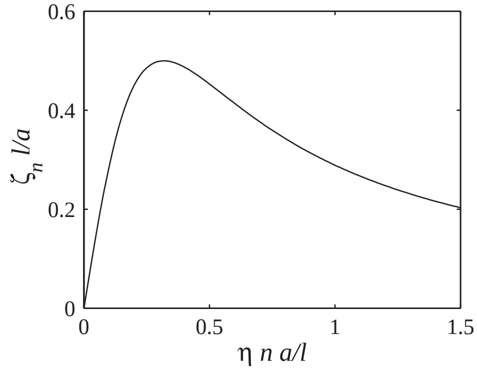

Fig. 2. The asymptotic modal damping relation.

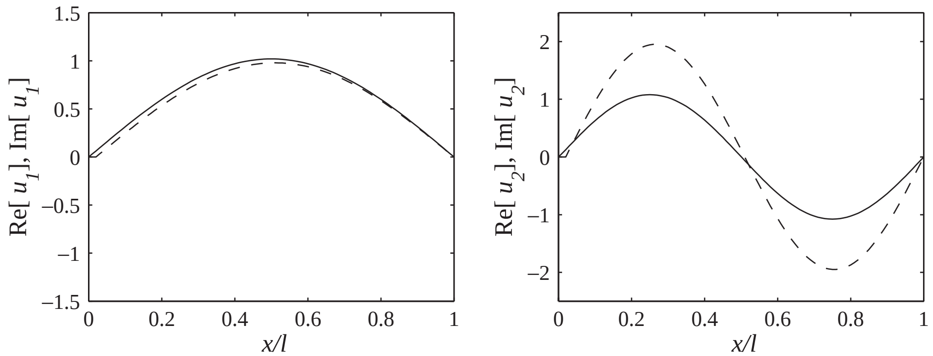

Fig. 3. Real and imaginary part of modes $u_1(x)$ and $u_2(x)$ for $a/\ell = 0.02$.

Thus, the higher modes require less damping to be optimal. However, the modal damping ratio represents the damping per oscillation period, and thus the damping per time unit may be more representative, when comparing the damping level of different modes. If the lowest mode has optimal damping the higher modes will have a damping ratio $\zeta_n \simeq (a/\ell)n/(1+n^2)$, corresponding the damping per time increment increasing with mode number as $n^2/(1+n^2)$. Thus, within the present assumption of $a \ll \ell/n$, the damping of the higher modes increase to double the value of the first mode with increasing mode number.

The two first complex modes are shown in Fig. 3 for $a/\ell = 0.02$ and optimal damping of the first mode, i.e. $\eta = (\pi a/\ell)^{-1}$. This provides a damping ratio of $\zeta_1 = 0.01$ for the first mode, deemed to be sufficient for cables in cable stayed bridges [12]. The modes are normalized by selecting $u_n(a) = \sin(\beta_{n,0}a)$, whereby the corresponding undamped mode would have maximum value equal to one. It is characteristic of all the modes that the real and imaginary part are similar and with the same number of nodes as in the undamped case. The effect of the damper shows up as a kink on the imaginary part at $x = a$. The first mode has optimal damping, and the real and imaginary part are approximately of the same magnitude apart from a neighborhood around the damper. For the higher modes the imaginary part becomes larger than the real part.

3.1.3. Response to local impulse

The modal mass M corresponding to the mode $u_n(x)$ given by Eq. (29) is calculated by use of the first of the integrals in Eq. (14) in order to avoid evaluating $\mathscr{D}_x u_n$ at the point of the damper. Integration gives

$$M_n = \frac{1}{2} m u_n(a)^2 \left\{ \frac{a}{\sin^2(\beta_n a)} \left[1 - \frac{\sin(2\beta_n a)}{2\beta_n a} \right] \right. \\ \left. + \frac{a'}{\sin^2(\beta_n a')} \left[1 - \frac{\sin(2\beta_n a')}{2\beta_n a'} \right] - \frac{\mathrm{i}}{\beta_n} \frac{c}{\sqrt{Tm}} \right\}. \tag{38}$$

When the second terms in the two brackets are reduced by introducing the product formula for the sine to the double angle, it follows from Eq. (31) that these terms cancel the last term due to the damper. Thus, the final form of the modal mass is

$$M_n = \frac{1}{2} m u_n(a)^2 \left\{ \frac{a}{\sin^2(\beta_n a)} + \frac{a'}{\sin^2(\beta_n a')} \right\}. \tag{39}$$

For undamped motion of the cable $u_n(a) = \sin(\beta_{n,0} a)$ leads to the modal mass $M_n = \frac{1}{2} m\ell$.

Let a concentrated force of magnitude F_b act at $x = b$ from $t = 0$. This gives the solution to the modal amplitude equation (18)

$$r_n(t) = -\frac{F_b u_n(b)}{2\lambda_n^2 M_n} (1 - \mathrm{e}^{\lambda_n t}). \tag{40}$$

The total response then follows by summation from Eq. (15). When the modes are grouped in conjugate pairs, the solution takes the form

$$u(x,t) = -F_b \sum_n \mathrm{Re}\left[\frac{u_n(x) u_n(b)}{\lambda_n^2 M_n} (1 - \mathrm{e}^{\lambda_n t}) \right]. \tag{41}$$

When considering the solution it is convenient to represent the time in terms of the time for a wave to transverse the cable. This time is determined by the wave speed $c_0 = \sqrt{T/m}$ and the cable length ℓ as ℓ/c_0. The static deflection is seen from the original equation (23) to be $\frac{1}{4} lF_b/T$.

The response is calculated by use of 20 mode shape pairs for a damper at $a/\ell = 0.02$ and a concentrated force F_b applied at the center, $b = \frac{1}{2}\ell$. The displacement at the center is shown in Fig. 4 for optimal damping of the first mode, $\eta = (\pi a/\ell)^{-1}$. The corresponding undamped solution is shown as a dashed curve. It is seen that in the case of damping the response at $x = b$ decreases rather rapidly. The reason is the damping in conjunction with a rounding of the shape as illustrated in Fig. 5 for the time $t = 5l/c_0$.

3.2. Beam with dampers

A simply supported homogeneous beam of length ℓ with bending stiffness EI and mass per unit length m is shown in Fig. 6. Damping is provided by rotation dampers at the supports, simulating flexible support in a structure. This problem was studied by Oliveto et al. [5], who developed a special iteration technique, and considered the response problem via an impulse formulation. In

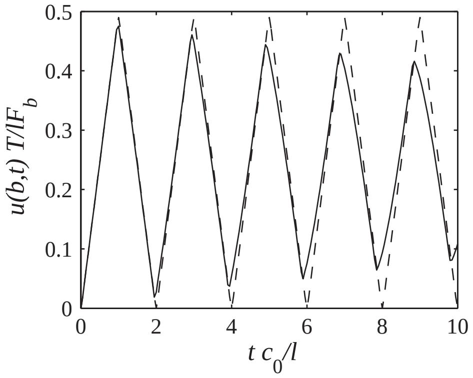

Fig. 4. Response history $u(b,t)$ for a force F_b at $b = \frac{1}{2}\ell$.

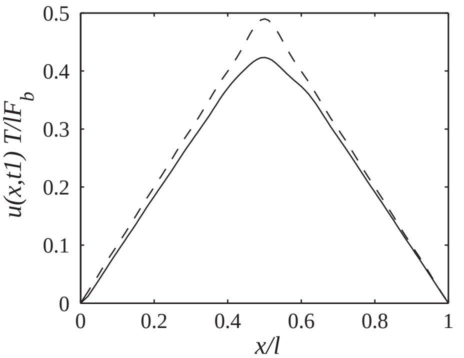

Fig. 5. Shape of cable at $t = 5\ell/c_0$.

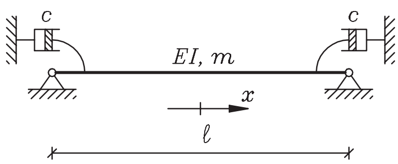

Fig. 6. Simply supported beam with viscous rotation dampers.

the following the problem is treated as an example of modal decomposition within the class of damped structural elements considered in Section 2, and optimal damper properties are derived by a simple generalization of the technique used in the cable example.

The beam in Fig. 6 is symmetric, and it is therefore convenient to introduce the x-co-ordinate with origin at the center of the beam. The beam is governed by a differential equation of type (4) with stiffness operator

$$\mathscr{D}_x u = (EIu'')'', \tag{42}$$

and damping operator

$$\mathscr{C}_x \dot{u} = -[c\delta(x+\tfrac{1}{2}\ell)\dot{u}']' - [c\delta(x-\tfrac{1}{2}\ell)\dot{u}']'. \tag{43}$$

The latter is perhaps best seen by formulating the rate of work by multiplication with a virtual velocity $\delta\dot{u}$. Both of these operators are self-adjoint, and the corresponding eigenvalue problem therefore symmetric.

The equation of motion of the beam with transverse load intensity $f(x,t)$ is

$$(EIu'')'' + m\ddot{u} = f(x,t). \tag{44}$$

When the solution $u(x,t)$ is represented in damped harmonic form (27) with complex angular frequency ω_n, the corresponding homogeneous equation is

$$(EIu_n'')'' - m\omega_n^2 u_n = 0. \tag{45}$$

The end moments of the beam equal the moments provided by the dampers,

$$M = -EIu'' = c\dot{u}' \quad \text{for } x = \pm\tfrac{1}{2}\ell. \tag{46}$$

The characteristic equation of this system permits solutions of the form

$$u_n(x) \propto \mathrm{e}^{\pm\beta_n x},\ \mathrm{e}^{\pm \mathrm{i}\beta_n x}, \tag{47}$$

where the wave number β_n is determined by

$$\beta_n^4 = \frac{m\omega_n^2}{EI}. \tag{48}$$

Thus, the representation of the present damped solution is similar to that of an undamped beam, when the frequency and thereby the wave number are permitted to take complex values.

A suitable non-dimensional damping parameter is identified from the boundary condition (46) by substitution of any of solutions (47). When the wave number and the frequency are eliminated by use of Eq. (48) the damping parameter is found to be

$$\eta = \frac{c/\ell}{\sqrt{mEI}}. \tag{49}$$

This definition of the non-dimensional damping is similar in form to Eq. (32) for the cable.

3.2.1. Symmetric modes

When the x-axis is located with origin at the center of the beam, the symmetric modes with $u_n(\pm\frac{1}{2}\ell) = 0$ can be constructed directly from a simple symmetry argument as

$$u_n(x) = C_n[\cos(\beta_n x)\cosh(\tfrac{1}{2}\beta_n\ell) - \cosh(\beta_n x)\cos(\tfrac{1}{2}\beta_n\ell)], \tag{50}$$

where C_n is a normalizing constant. After differentiation, substitution of this mode into the boundary condition (46) yields the following equation for the wave number β_n

$$\cot(\tfrac{1}{2}\beta_n\ell) = -\mathrm{i}\eta(\tfrac{1}{2}\beta_n\ell)[1+\tanh(\tfrac{1}{2}\beta_n\ell)\cot(\tfrac{1}{2}\beta_n\ell)]. \tag{51}$$

The undamped modes correspond to the roots $\beta_{n,0}\ell = n\pi$, $n = 1,3,5,\ldots$. This suggests that all terms containing $\cot(\frac{1}{2}\beta_n\ell)$ are collected at the left of the equation, which after division by the

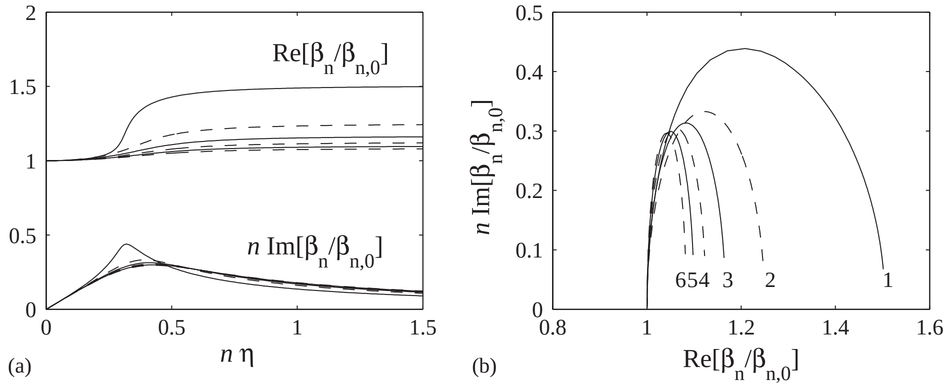

Fig. 7. Wave number β_n, (a) as function of damping, (b) in complex plane.

factor of the cot-function takes the form

$$\cot(\tfrac{1}{2}\beta_n\ell) = \frac{-\mathrm{i}\eta(\frac{1}{2}\beta_n\ell)}{1+\mathrm{i}\eta(\frac{1}{2}\beta_n\ell)\tanh(\frac{1}{2}\beta_n\ell)}, \quad n = 1,3,5,\ldots \quad (52)$$

This formula is very similar to the wave number equation (33) for the taut string with a concentrated damper, and can be used directly for iteration, starting with the undamped wave number $\beta_{n,0}\ell = n\pi$ for each of the symmetric modes $n = 1,3,5,\ldots$.

The results are shown in Fig. 7, where the wave number of the first three symmetric modes is shown in full line. Note that in order to fit the graphs of all the six lowest modes into the same figures the real part of all the wave numbers are normalized with their corresponding undamped value, $\mathrm{Re}[\beta_n]/\beta_{n,0}$, while the imaginary part is normalized by the undamped wave number of the first mode, $\mathrm{Im}[\beta_n]/\beta_{1,0}$. It is seen from Fig. 7a that the real part of the wave number increases from its initial undamped value $\beta_{n,0} = n\pi/\ell$ to a value $\beta_{n,\infty}$, corresponding to clamped end conditions. The wave numbers $\beta_{n,\infty}$ of the clamped beam follows from Eq. (52) as the limit for infinite damping, $\eta \to \infty$,

$$\cot(\tfrac{1}{2}\beta_{n,\infty}\ell) = -\coth(\tfrac{1}{2}\beta_{n,\infty}\ell), \quad n = 1,3,5,\ldots \quad (53)$$

The increase mainly takes place over the interval of damping $0.3 \leqslant n\eta \leqslant 0.5$. The imaginary part of the wave number increases from zero, reaches a peak for damping in the interval $0.3 \leqslant n\eta \leqslant 0.5$, and then decreases towards zero again.

3.2.2. Anti-symmetric modes

The anti-symmetric modes with $u(\pm\frac{1}{2}\ell) = 0$ are treated in a similar way. The anti-symmetric solutions can be written down directly as

$$u_n(x) = C_n[\sin(\beta_n x)\sinh(\tfrac{1}{2}\beta_n\ell) - \sinh(\beta_n x)\sin(\tfrac{1}{2}\beta_n\ell)], \quad (54)$$

with normalizing constant C_n. Differentiation and substitution into the boundary condition (46) leads to the wave number equation for the anti-symmetric modes,

$$\tan(\tfrac{1}{2}\beta_n\ell) = \mathrm{i}\eta(\tfrac{1}{2}\beta_n\ell)[1 - \coth(\tfrac{1}{2}\beta_n\ell)\tan(\tfrac{1}{2}\beta_n\ell)]. \quad (55)$$

The wave numbers of the anti-symmetric undamped modes $\beta_{n,0}\ell = n\pi$ for $n = 2, 4, 6, \dots$ are the roots of $\tan(\frac{1}{2}\beta_n\ell)$, and the terms containing the tan-function are collected on the left side, whereby the final equation becomes

$$\tan(\tfrac{1}{2}\beta_n\ell) = \frac{\mathrm{i}\eta(\frac{1}{2}\beta_n\ell)}{1 + \mathrm{i}\eta(\frac{1}{2}\beta_n\ell)\coth(\frac{1}{2}\beta_n\ell)}, \quad n = 2, 4, 6, \dots \; . \tag{56}$$

Also this equation is suitable for direct iteration, starting with $\beta_{n,0}\ell = n\pi$ for each of the anti-symmetric modes $n = 2, 4, 6, \dots$. The limiting behaviour of infinite damping gives the following equation for the wave numbers of the anti-symmetric modes of a clamped beam,

$$\tan(\tfrac{1}{2}\beta_{n,\infty}\ell) = \tanh(\tfrac{1}{2}\beta_{n,\infty}\ell), \quad n = 2, 4, 6, \dots \; . \tag{57}$$

This equation is identical to Eq. (53) for the symmetric modes, apart from a change of the sign of the right side of the equation.

The results for the first three anti-symmetric modes are included in Fig. 7 in dashed line. The results for the anti-symmetric modes fit in between the previous and following symmetric modes.

3.2.3. *Optimal damping and tuning*

Damping is determined by the complex damped natural frequency ω_n. It follows from relation (48) that $\omega_n \propto \beta_n^2$, and the frequency curves corresponding to the wave number curves in Fig. 7 are shown in Fig. 8. It is seen that by scaling the damping of the individual modes as $n\eta$ and by scaling the imaginary part of the frequency as $n\,\mathrm{Im}[\omega_n/\omega_{n,0}]$ the curves approach a common curve for the higher modes. Although the results for the beam show the same structure as those for the cable they are not quite as simple analytically. This is due to the fact that relation (34) for the increase of wave number as a consequence of damping contains the small factor a/ℓ. This implies that the change in wave number is small, and thus the original undamped wave number can be used in the right hand side of the relation. In the case of beams the similar wave number relations (52) and (56) do not contain a small factor. Thus, in the case of beams large changes of the wave number may occur, particularly for the lower modes, and thus use of the undamped wave number on the right hand side of the relation is not a generally good approximation.

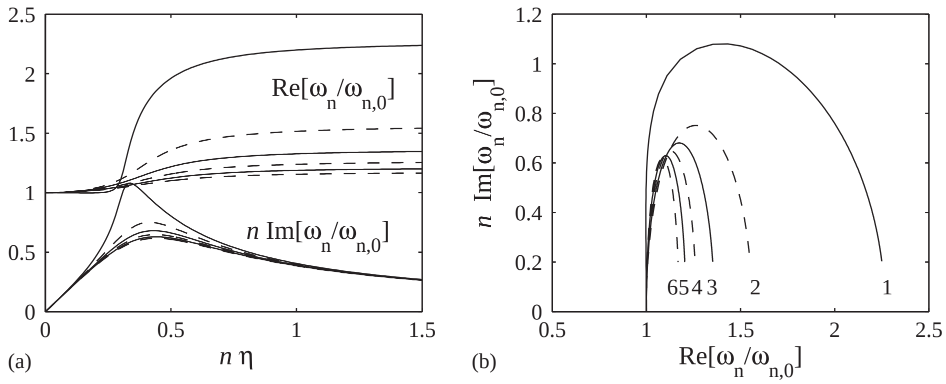

Fig. 8. Angular frequency ω_n, (a) as function of damping, (b) in complex plane.

Table 1
Complex frequency at optimal damping

n	1	2	3	4	5	6
$n\eta_{opt}$	0.377	0.415	0.432	0.439	0.442	0.444
$\mathrm{Re}[\omega_n/\omega_{n,0}]$	1.403	1.262	1.175	1.131	1.104	1.086
$n\,\mathrm{Im}[\omega_n/\omega_{n,0}]$	1.081	0.751	0.681	0.648	0.630	0.617

Optimal damping of the individual modes is obtained when the imaginary part of the damped frequency $\mathrm{Im}[\omega_n]$ attains its maximum value. Corresponding values of $\mathrm{Im}[\omega_n]_{\max}$ and the damping parameter η_{opt} are given for the first six modes in Table 1. As in the case of the cable optimal damping of the first mode will lead to similar attenuation rate of the higher modes. However, in the case of beams the damping is much higher and optimal tuning of the dampers may therefore be less critical.

4. Conclusions

It has been demonstrated that viscous damping and isolated dampers can be included in the equations of motion in a convenient operator format that permits a general formulation of the complex-valued damped eigenvalue problem. In the absence of gyroscopic forces a self-adjoint damping operator leads to a symmetric damped eigenvalue problem in which the complex vibration modes satisfy two sets of orthogonality relations.

The theory is used to illustrate the role of complex damped modes in vibrations of cables and beams. It is demonstrated that for concentrated dampers the wave number equations are similar for cables and beams, and a simple iterative solution method is presented. Properties of the damped modes are used to identify the maximum obtainable damping and the corresponding optimal tuning of the damper. Furthermore the role of the damped complex modes in defining the modal loads is demonstrated in the case of a taut cable with an isolated damper.

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References

[1] K.A. Foss, Coordinates which uncouple the equations of motion of damped linear dynamic systems, *Journal of Applied Mechanics* 35 (1958) 361–367.
[2] M. Geradin, D. Rixen, *Mechanical Vibrations*, 2nd Edition, Wiley, Chichester, 1997.
[3] R. Singh, W.M. Lyons, G. Prater, Complex eigensolution for longitudinally vibrating bars with a viscously damped boundary, *Journal of Sound and Vibration* 133 (1989) 364–367.

[4] A.J. Hull, A closed form solution of a longitudinal bar with a viscous boundary layer, *Journal of Sound and Vibration* 169 (1994) 19–28.
[5] G. Oliveto, A. Santini, E. Tripodi, Complex modal analysis of a flexural vibrating beam with viscous end conditions, *Journal of Sound and Vibration* 200 (1997) 327–345.
[6] B. Yang, X. Wu, Transient response of one-dimensional distributed systems: a closed form eigenfunction expansion realization, *Journal of Sound and Vibration* 208 (1997) 763–776.
[7] B.N. Pacheco, Y. Fujino, A. Sulekh, Estimation curve for modal damping in stay cables with viscous damper, *Journal of Structural Engineering* 119 (1993) 1961–1979.
[8] S. Krenk, Vibrations of a taut cable with an external damper, *Journal of Applied Mechanics* 67 (2000) 772–776.
[9] J.A. Main, N.P. Jones, Free vibrations of taut cable with attached damper. I: Linear viscous damper, *Journal of Engineering Mechanics* 128 (2002) 1062–1071.
[10] S. Krenk, S.R.K. Nielsen, Vibrations of a shallow cable with a viscous damper, *Proceedings of the Royal Society, London, Series A* 458 (2002) 339–357.
[11] S.R.K. Nielsen, S. Krenk, Whirling motion of a shallow cable with viscous dampers, *Journal of Sound and Vibration* 265 (2003) 417–435.
[12] C. Geurts, T. Vrouwenvelder, P. van Staalduinen, J. Reusink, Numerical modelling of rain-wind-induced vibration: Erasmus Bridge, Rotterdam, *Structural Engineering International* 2 (1998) 129–135.